Advanced Modern Macroeconomics

Monetary Theory and Policy

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- Introduces inflation, builds fundamentals of monetary economics,
  - including application to central bank policy.
  - Starts with main link between real, and "nominal" economy:
  - Market interest rates equal real interest rate plus inflation rate.
- How inflation is created by financing government expenditure
  - by printing money.
  - Relation of money supply growth rate, inflation rate,
  - fraction of output spent by government, financed by printing money.
  - Hyperinflation around world explained.

Monetary Models of Exogenous, Endogenous Growth

- Formal monetary economy:
  - inflation as tax falls on purchases,
  - induces substitution from goods to leisure, less employment.
  - Similar to goods, labor taxes;
  - physical capital to human capital ratio rises.
- With endogenous growth, inflation tax reduces growth rate.
- Central bank policy: derive equilibrium "Taylor rule" of interest rates,
  - given money supply growth determination.
  - Financial crisis: role of policy subsidizing/taxing real interest rate.

Building on the Last Chapters

- Chapter 18 introduced price level, money supply,
  - only in government budget constraint,
  - without representative agent optimization.
- This chapter how increased money supply growth
  - increases inflation, distorts margins and
  - taxes economy as in Parts 4 and 5.
- Chapter 9, 19 introduce taxes
  - into dynamic exogenous growth, endogenous growth economies.
  - This chapter does same: for inflation tax.
  - Not an explicit tax like labor income, capital income, goods taxes,
  - Yet inflation affects economy exactly like explicit tax.
- Tax analysis very similar to Chapters 9, 19..

Learning Objective

- Monetary economy incorporates money supply and price level
    - so as to reduce down to standard economy with a tax.
    - Inflation tax affects marginal rate of substitution; goods, leisure.
- Financing government expenditure without regular explicit taxes,
    - but by central bank money printing, buy government bonds,
    - creates inflation tax results.
- Making economy a monetary one shows effect of inflation tax.

Who Made It Happen

- Bailey 1956 "The Welfare Cost of Inflationary Finance"
  - started modern literature in formalizing inflation tax effects.
  - Cagan 1956: same framework to explain hyperinflation across Europe.
- Samuelson 1958, Sidrauski 1967, Lucas 1975:
  - inflation in general equilibrium dynamic models.
- Lucas 1980, Goodfriend, McCallum 1987:
  - Money used for exchange, in dynamic model.
- Gomme 1993, Gillman, Kejak 2005:
  - money for exchange in endogenous growth;
  - inflation tax reduces economic growth.

The Fisher Equation: Linking Nominal and Real

- Irving Fisher 1896 Theory of the Interest Rate:
  - market interest rate comprised of real and nominal component.
  - Cornerstone of link in macroeconomics between monetary and real sides.
  - Here derived in dynamic baseline economy by including second asset,
  - and nominal prices.
- Consumer invests in physical capital, and government bonds.
  - Nominal bonds in terms of current dollar units.
  - Dividing nominal bonds by price of goods, gives real bonds.
  - Account for price level in dynamics equation introduces inflation rate,
  - into return on nominal bonds.
- Inflation rate partly determine bond return,
  - links real and nominal sides.

Nominal versus Real Notation

- Let P_t be "nominal" price of goods.
  - Usually $P_t = 1$ and is suppressed, focusing on real side.
  - Relative price of goods is just $\frac{1}{w_t}$,
- Nominal price divided by nominal wage: same result,
  - $\frac{P_t}{P_t w_t} = \frac{1}{w_t}$.
- Market interest rate on government bonds still R_t.
- Nominal government bonds denoted by B_t
  - "real" government bonds: $\frac{B_t}{P_t} \equiv b_t$.
- Add government budget constraint:

$$P_t G_t = B_{t+1} - B_t \left(1 + R_t\right),$$

  - with real spending G_t given to consumer, financed by bonds.

Deriving the Fisher Equation

- Nominal budget constraint:

$$\begin{aligned} P_t c_t &= P_t w_t l_t + P_t r_t k_t - P_t k_{t+1} + P_t k_t (1-\delta_k) \\ &\quad - B_{t+1} + B_t (1+R_t) + P_t G_t; \\ c_t &= w_t l_t + r_t k_t - k_{t+1} + k_t (1-\delta_k) \\ &\quad - \frac{B_{t+1}}{P_t} + \frac{B_t}{P_t}(1+R_t) + G_t. \end{aligned}$$

- Denote gross inflation rate: $1+\pi_t$; define as ratio of price levels:

$$\frac{P_{t+1}}{P_t} \equiv 1 + \pi_{t+1}.$$

$$\begin{aligned} \frac{B_{t+1}}{P_t} &= \frac{B_{t+1} P_{t+1}}{P_{t+1} P_t} = b_{t+1}(1+\pi_{t+1}); \\ c_t &= w_t l_t + r_t k_t - k_{t+1} + k_t (1-\delta_k) \\ &\quad - b_{t+1}(1+\pi_{t+1}) + b_t (1+R_t) + G_t. \end{aligned}$$

Consumer Optimization with Inflation and Bonds

$$V\left(k_t, b_t\right) = \underset{l_t, k_{t+1}, b_{t+1}}{Max}$$

$$\ln\left[w_t l_t - k_{t+1} + k_t\left(1 + r_t - \delta_k\right) - b_{t+1}\left(1 + \pi_{t+1}\right) + b_t\left(1 + R_t\right) + G_t\right]$$
$$+\alpha \ln\left(1 - l_t\right) + \beta V\left(k_{t+1}, b_{t+1}\right);$$

$$\Longrightarrow x_t = \frac{\alpha c_t}{w_t},\ \frac{c_{t+1}}{\beta c_t} = 1 + r_t - \delta_k.$$

- Equilibrium conditions with respect to b_{t+1} and b_t :

$$\frac{1}{\beta c_t}\left(1 + \pi_{t+1}\right) = \frac{\partial V\left(k_{t+1}, b_{t+1}\right)}{\partial b_{t+1}},\ \frac{\partial V\left(k_t, b_t\right)}{\partial b_t} = \frac{1}{c_t}\left(1 + R_t\right);$$

The Fisher Equation

- Bond equilibrium conditions imply

$$\Longrightarrow \quad \frac{1}{\beta}\frac{1}{c_{t-1}}\left(1+\pi_t\right)=\frac{1}{c_t}\left(1+R_t\right)=0,$$

$$1+R_t = \frac{c_t}{\beta c_{t-1}}\left(1+\pi_t\right).$$

- Combined with intertemporal condition from capital:

$$\frac{c_{t+1}}{\beta c_t}=1+r_t-\delta_k,$$

- Gives the Fisher equation:

$$\begin{aligned}
1+R_t &= \frac{c_t}{\beta c_{t-1}}\left(1+\pi_t\right)=\left(1+r_t-\delta_k\right)\left(1+\pi_t\right),\\
1+R_t &= \left(1+r_t-\delta_k\right)\left(1+\pi_t\right).\\
ln\left(1+x\right) &\simeq x;\\
&\Longrightarrow R_t=r_t-\delta_k+\pi_t.
\end{aligned}$$

Understanding the Fisher Equation

- Higher is inflation rate, higher is nominal interest rate.
- Investment that yields a real return of 3% in one year
    - must account for change in aggregate price level P_t over year.
    - If inflation 2%, market, or nominal, interest rate must be 5%
    - to yield 3% real interest rate yield.
- Unexpected increase in inflation lowers ex-post real return.

Unexpected Inflation and Negative Real Returns

- Fisher equation: both short run and long run effects,
  - as result of inflation, changes in level of inflation.
  - Depends if inflation properly anticipated, or expected,
  - versus unanticipated inflation change.
- May expect inflation rate π_t at time t.
  - But unexpected government expenditure can increase inflation;
  - as during Vietnam War of 1970s, or Iraq-Afghanistan Wars of post 2001,
  - post bank crisis bail-out 2007-2010.
- If not totally anticipated, higher inflation lowers real return.
- Negative real interest rates experienced in US 1970s
  - during 2009-2011, in US, UK, EU.

The Phillips Curve

- Unanticipated inflation can cause a "Phillips curve",
  - whereby accelerating inflation causes higher output,
  - decelerating inflation causes lower output.
- Link to real interest rate of Fisher equation:
  - negative real rate decreases investment cost, may stimulate output;
  - higher real interest rate because of unexpected decrease in inflation,
  - causes output to fall in short run.
- Rise in output from unexpected acceleration of inflation
  - basic concept behind Phillips curve.
  - Statistically relation found during many periods,
  - especially when inflation accelerating or decelerating.
- Harder to empirically identify Phillips relation
  - when inflation changing as anticipated,
  - such as post 1983 US; Stock, Watson, 1999.

Interpreting the Phillips Curve Relation

- Higher output from unanticipated inflation not efficient.
  - Consumers, firms, borrowers, lenders, making mistakes.
  - Inefficiency caused by variation, or volatility, in inflation.
  - Causes risk premium for inflation, increases real interest rate.
- Governments not wise to try to exploit inflation-output Phillips curve.
  - Once anticipated, must accelerate inflation even more.
  - Just adds inflation volatility, inefficiency.
  - Only lasts in short run and not over long run.
- Inefficiency from Phillips curve trade-off is long run result.

Money supply, Inflation and Government Expenditure

- Long historical link established empirically
  - between inflation rate and money supply growth rate.
- Show link by assuming government use new money creation
  - to finance some fraction of new government expenditure.
- Denote new expenditure in real terms by G_t^n.
  - Assume BGP equilibrium,
  - stationary rate of money supply growth, denoted by σ_t :

$$\frac{M_{t+1} - M_t}{M_t} = \sigma_{t+1}.$$

- Government spending equal to real value of newly printed money:

$$G_t^n = \frac{M_{t+1} - M_t}{P_t}.$$

Money Supply Growth and Government Spending

- Let government spending financed by money
  - be a fraction of ouput η :

$$\frac{G_t^n}{y_t} = \eta.$$

  - Combining with finance equation through money,

$$\frac{G_t^n}{y_t} = \eta = \frac{M_{t+1} - M_t}{P_t y_t}.$$

  - Can write in terms of money supply growth rate:

$$\eta = \frac{M_{t+1} - M_t}{y_t} = \frac{M_{t+1} - M_t}{M_t} \frac{M_t}{P_t y_t} = \frac{\sigma_{t+1} M_t}{P_t y_t}.$$

Money Supply Growth, Spending, Velocity

- Solution of money supply growth rate:

$$\sigma_{t+1} = \eta \frac{P_t y_t}{M_t}.$$

  - Simplifgy by writing money stock M_t in real terms:

$$m_t \equiv \frac{M_t}{P_t}.$$

  - $$\Longrightarrow \sigma_{t+1} = \eta \frac{y_t}{m_t}.$$

Money Supply Growth, Spending, Velocity

- $\frac{y_t}{m_t}$: income velocity of money is output to money ratio.
  - If $\frac{y_t}{m_t} = 1$ all output bought with money;
  - money supply growth rate equals government expenditure share:

$$\sigma_{t+1} = \eta \frac{y_t}{m_t} = \eta.$$

- Usually output bought with money and credit,
  - velocity greater than one,
  - money supply growth rate exceeds government expenditure share:

$$\sigma_{t+1} = \frac{\eta y_t}{m_t} > \eta.$$

- Real money holdings depends on inflation rate;
  - this is money demand theory.
  - Higher inflation rate, less real money, higher velocity;
  - higher is σ needed to finance given η.
  - Money printed at faster rate to purchase same expenditure,
  - if $\frac{y_t}{m_t}$ high.

Expenditure Share, Money Supply Growth, Inflation

- Instead of money supply growth rate,
  - Write expenditure share in terms of inflation, growth rate:

$$
\begin{aligned}
\frac{G_t^n}{y_t} &= \eta = \frac{M_{t+1} - M_t}{P_t y_t} = \frac{M_{t+1} P_{t+1}}{P_{t+1} P_t y_t} - \frac{M_t}{P_t y_t} \\
&= \frac{m_{t+1} \left(1 + \pi_{t+1}\right)}{y_t} - \frac{m_t}{y_t}.
\end{aligned}
$$

  - Possible output growth whereby

$$
\begin{aligned}
\frac{y_{t+1}}{y_t} &= 1 + g_{t+1}, \\
\eta &= \frac{m_{t+1} y_{t+1} \left(1 + \pi_{t+1}\right)}{y_{t+1} y_t} - \frac{m_t}{y_t} \\
&= \frac{m_{t+1}}{y_{t+1}} \left(1 + g_{t+1}\right) \left(1 + \pi_{t+1}\right) - \frac{m_t}{y_t}
\end{aligned}
$$

Inflation Rate Determination with Zero Growth

- Assume $g = 0$ along BGP

$$
\begin{aligned}
\frac{m_{t+1}}{m_t} &= \frac{y_{t+1}}{y_t} = 1 + g, \\
&\Longrightarrow \frac{m_{t+1}}{y_{t+1}} = \frac{m_t}{y_t} = \frac{m}{y}, \\
\eta &= \frac{m_{t+1}}{y_{t+1}} \left(1 + g_{t+1}\right) \left(1 + \pi_{t+1}\right) - \frac{m_t}{y_t}, \\
\eta &= \frac{m}{y} (1 + \pi) - \frac{m}{y} = \pi \frac{m}{y}; \\
&\Longrightarrow \pi = \eta \frac{y}{m}.
\end{aligned}
$$

- Inflation equals expenditure share factored by velocity;
  - Higher is velocity, for given η,
  - higher must be inflation.

Money Supply Growth, Inflation, Expenditure Share Equality

- Combined money supply growth rate condition,inflation conditions

$$\begin{aligned} \sigma_{t+1} &= \eta \frac{y_t}{m_t}, \\ \pi &= \eta \frac{y}{m} = \sigma. \end{aligned}$$

- Money supply growth rate equals inflation rate.
- Further, if velocity is one,

$$\begin{aligned} m &= y, \\ \sigma &= \pi = \eta \frac{y}{m} = \eta. \end{aligned}$$

- Expenditure share, money supply growth, inflation rates all equal:

$$\pi = \eta = \sigma.$$

- Finance share directly implies inflation and money supply growth.

Example 20.1 Financing High Spending when Zero Growth

- Assume $\eta = 10\%$, $g = 0$; then $\pi = \sigma$.
  - But need to know velocity of money to determine level of π, σ.
  - Assume velocity given at $\frac{y}{m} = 5$.

$$\Longrightarrow \sigma = \pi = \eta \left(\frac{y}{m} \right) = (0.10)\, 5 = 0.5.$$

- Very high inflation, money supply growth rate of 50%;
- plausible if large percent of output purchased by printing money.
  - Similar rates seen in post-Soviet economies after 1990
  - when new governments had few direct tax revenue sources.
  - 50% inflation rate is border of what is called hyperinflation.

Example 20.2 Financing Low Spending when Zero Growth

- Assume $\eta = 2\%$, g=0; assume velocty $\frac{y}{m} = 2$;

$$\eta \left(\frac{y}{m} \right) = (0.02)\,2 = 0.04 = \sigma = \pi.$$

- Money supply growth rate and inflation rate of 4%
  - finance 2% of output in zero growth economy.
  - Plausible type of "normal" levels of inflation tax revenue
- Seen in Western governments under low "inflation targets".

Collecting the Inflation Tax

- When government prints new money it collects inflation tax.
  - Government new money to buy output,
  - prices rise because of greater money supply,
  - less real goods can be bought by consumer with same money.
  - Inflation rises as price level rises, and money worth less;
  - money is taxed by inflation, decreasing its value.
- Two parts to inflation tax:
  - its collection as government prints new money, spends it;
  - how it imposes upon consumer by reducing value of existing money:
  - real expenditure consumer budget is decreased by inflation tax.
- Velocity enters into determination of inflation level.
  - If less money used by consumer, higher must be inflation tax rate
  - to raise same amount of revenue for constant share η.
  - Broader is "tax base", lower can be tax rate, for given η.
  - But tax base size is real money demand: depends in turn on π.

Inflation and Growth

- With BGP $g > 0$,

$$\eta = \frac{m}{y}(1+g)(1+\pi) - \frac{m_t}{y_t};$$
$$\pi = \frac{\frac{\eta y}{m} - g}{(1+g)}.$$

  - Inflation rate smaller, for given η, given velocity, higher is g.
- Growing economy has greater demand for money,
  - broader inflation tax base over time, lower inflation rate
  - to finance output fraction η.
- For given money supply growth rate σ, with $g > 0$,
  - again, inflation rate is lower:

$$\pi = \frac{\frac{\eta y}{m} - g}{(1+g)} = \frac{\sigma - g}{(1+g)}.$$

  - Feedback possible between inflation rate, growth rate:
  - inflation tax on goods causes more leisure, less output growth.

Inflation Tax Dynamics and Hyperinflation

- If consumer holds less real money when inflation goes up,
  - velocity $\frac{y}{m}$ increases; money supply growth rate σ
  - for paying η share of government expenditure must rise.
  - As σ rises, so does inflation rate π again rise.
  - If $\frac{y}{m}$ increases further, then again σ must rise, and π again rises.
  - Can be upward spiral into hyperinflation, with extremely inflation π.
- Cause of hyperinflation:
  - government pays for expenditure by printing money,
  - consumer does not want to hold real money
  - sufficient to provide tax base from which to raise inflation tax.
- So government raises moneys supply growth σ,
  - inflation π rate even higher, but consumer increases money velocity,
  - by holding less real money.
  - Upward spiral of money creation, inflation until
  - government collapses or currency re-established,
  - with re-capitalized treasury that does not print much money.

Inflation Dynamics

- Solve for next period inflation from government share η :

$$\eta = \frac{m_{t+1}}{y_{t+1}} (1 + g_{t+1}) (1 + \pi_{t+1}) - \frac{m_t}{y_t},$$
$$\pi_{t+1} = \eta \frac{y_{t+1}}{m_{t+1} (1 + g_{t+1})} + \frac{m_t}{m_{t+1}} - 1.$$

  - Increase in money velocity $\frac{y_{t+1}}{m_{t+1}}$,
  - decrease in growth rate of output g_{t+1},
  - decrease in real money demand m_{t+1},
  - all cause higher inflation rate.

- During recession,
  - growth rate of output falls, inflation falls.
  - Velocity of money often falls in recessions, decreasing inflation rate.
  - Real money demand may fall, pressure inflation rate upwards.

Government Spending and Money Supply

- Rise in government expenditure share rises η
  - again financed by new money:
  - inflation rate rises.
  - Even higher inflation rate if velocity rises, growth rate falls.
- During war or business cycle crisis,
  - government often increase government expenditure
  - and its financing by printing more money.
  - Some evidence for this in US 1970s, 1980s, during Vietnam War.
  - And recent evidence during 2007-2010 financial crisis.
- During hyperinflation: rise in η financed by new money.

Hyperinflation in Europe and across the World

- During hyperinflation
  - money supply growth, inflation rate rise exponentially;
  - real money demand falls steadily; velocity rises steadily.
- Ends when government collapses or/and new currency instituted.
- Hyperinflation after World War I, World War II,
  - typically for defeated nations: insufficient taxable income base.
  - After WWI, in 1920s: Austria, Germany, Hungary, Poland, Russia;
  - during WWII: Greece, after WWII: Hungary.
- Other examples:
  - Latin America: Argentina, Chile in 1970s;
  - Bolivia, Brazil, Nicaragua Peru in 1980s, 1990s;
  - Africa: Zaire, Zimbabwe in 1990s, presently;
  - Eastern Europe: after Soviet Union demise in 1990s;
  - Belarus, Bulgaria, Georgia, Romania, Russia, Ukraine.

- Government finances by printing money, issuing bonds.
  - Denote nominal government expenditure by G_t :

$$G_t = M_{t+1} - M_t + B_{t+1} - B_t (1 + R_t)$$

  - Denote rate of growth of money supply by σ_t at time t :

$$\sigma_t \equiv \frac{M_{t+1} - M_t}{M_t}.$$

  - Assume money supply rate σ_t : given to consumer as exogenous.
- Cost of using money is interest not earned while holding money,
  - instead of investing in government bonds.

Benefit of Using Money: Exchange Constraint

- Benefit of money: need it to buy goods,
  - Assume "exchange constraint", or exchange technology:
  - only consumption goods purchased by money:

$$M_t = P_t c_t.$$

- Consumption velocity of money is $\frac{c_t}{m_t} = 1$.
  - Basic model: velocity can be endogenized as extension;
  - this model captures inflation tax increasing shadow price of goods
  - within marginal rate of substitution between goods and leisure.
- Extension with endogenous velocity left for advanced work,
  - allowing focus on mechanism of inflation tax.
  - Endogenizing velocity gives way to avoid inflation tax,
  - makes burden of inflation tax less,
  - by using banking credit instead of money use.

Consumer Budget Constraint

- From constraint with bonds,

$$\begin{aligned} c_t &= w_t l_t + r_t k_t - k_{t+1} + k_t \left(1 - \delta_k\right) \\ &\quad - b_{t+1}\left(1 + \pi_{t+1}\right) + b_t \left(1 + R_t\right) + G_t, \end{aligned}$$

  - multiply by price P_t, subtract investment in nominal money of $-M_{t+1} + M_t$:

$$\begin{aligned} P_t c_t &= P_t w_t l_t + P_t r_t k_t - P_t k_{t+1} + P_t k_t \left(1 - \delta_k\right) \\ &\quad - P_t b_{t+1}\left(1 + \pi_{t+1}\right) + P_t b_t \left(1 + R_t\right) - M_{t+1} + M_t + \frac{G_t}{P_t}. \end{aligned}$$

  - Note: money earns no interest over time,
  - bonds earn $1 + R_t$ gross interest; shows cost of holding money.

Consumer Budget Constraint

- Again in real terms using that $m_t \equiv \frac{M_t}{P_t}$:

$$\begin{aligned} c_t &= w_t l_t + r_t k_t - k_{t+1} + k_t \left(1 - \delta_k\right) - b_{t+1} \left(1 + \pi_{t+1}\right) \\ &\quad + b_t \left(1 + R_t\right) - \frac{M_{t+1}}{P_t} + \frac{M_t}{P_t} + \frac{G_t}{P_t}; \\ &= w_t l_t + r_t k_t - k_{t+1} + k_t \left(1 - \delta_k\right) - b_{t+1} \left(1 + \pi_{t+1}\right) \\ &\quad + b_t \left(1 + R_t\right) - m_{t+1} \left(1 + \pi_{t+1}\right) + m_t + \frac{G_t}{P_t} \end{aligned}$$

Use Budget Constraint to Solve for Leisure in Utility

- Solve for labor l_t, from budget constraint:

$$\begin{aligned}
l_t &= \frac{c_t}{w_t} - \frac{r_t k_t - k_{t+1} + k_t (1-\delta_k)}{w_t} - \frac{-b_{t+1}(1+\pi_{t+1})}{w_t} \\
&\quad - \frac{b_t(1+R_t)}{w_t} - \frac{-m_{t+1}(1+\pi_{t+1}) + m_t + \frac{G_t}{P_t}}{w_t}; \\
c_t &= m_t; \\
l_t &= \frac{m_t}{w_t} - \frac{r_t k_t - k_{t+1} + k_t (1-\delta_k)}{w_t} - \frac{-b_{t+1}(1+\pi_{t+1})}{w_t} \\
&\quad - \frac{b_t(1+R_t)}{w_t} - \frac{-m_{t+1}(1+\pi_{t+1}) + m_t + \frac{G_t}{P_t}}{w_t}.
\end{aligned}$$

- Leisure is $1 - l_t$, substitute into utility function.

$$
\begin{aligned}
& V\left(k_t, b_t, m_t\right) \\
= \quad & \underset{k_{t+1}, b_{t+1}, m_{t+1}}{Max} : \ln\left(m_t\right) + \alpha \ln\left[1 - \frac{r_t k_t - k_{t+1} + k_t\left(1-\delta_k\right)}{w_t}\right. \\
& \left. - \frac{-b_{t+1}\left(1+\pi_{t+1}\right) + b_t\left(1+R_t\right)}{w_t} - \frac{-m_{t+1}\left(1+\pi_{t+1}\right) + m_t + \frac{G_t}{P_t}}{w_t}\right] \\
& + \beta V\left(k_{t+1}, b_{t+1}, m_{t+1}\right);
\end{aligned}
$$

- Equilibrium conditions, envelope conditions
  - now include inflation tax, along BGP, goods-leisure margin:

$$
\frac{\alpha c_t}{x_t} = \frac{w_t}{1+R_t}.
$$

Basic Distortion of Monetary Economy

- Instead of standard

$$\frac{\alpha c_t}{x_t} = w_t,$$

  - now

$$\frac{\alpha c_t}{x_t} = \frac{w_t}{1 + R_t}.$$

  - Higher is nominal interest R_t,
  - more leisure taken, less consumption.
- Only if $R_t = 0$, no distortion, as in non-monetary economy
  - $R_t > 0$, inflation tax in effect positive.
- Inflation tax defined as level of R_t,
  - rather than inflation rate π_t itself
  - Opportunity cost of money to buy goods, is R_t;
  - also level of inflation tax in economic sense.

Nominal Interest Rate as Level of Inflation Tax

- Intertemporal marginal rate of substitution unaffected

$$1 + g_{t+1} = \frac{c_{t+1}}{c_t} = \frac{1 + r_{t+1} - \delta_k}{1 + \rho}.$$

- Fisher equation of interest rates holds:

$$1 + R_t = (1 + r_t - \delta_k)(1 + \pi_t).$$

- And nominal interest rate can be solved as function of
  - exogenous money supply growth rate σ,
  - time preference parameter ρ.

Solution of Interest Rate with Zero Growth

- Assume BGP growth rate $g = 0$,

$$
\begin{aligned}
1+g &= 1 = \frac{1+r_t-\delta_k}{1+\rho} \\
r_t-\delta_k &= \rho; \\
1+R_t &= (1+r_t-\delta_k)(1+\pi_t), \\
1+R_t &= (1+\rho)(1+\pi_t).
\end{aligned}
$$

Inflation Rate in Terms of Money Supply Growth Rate

- Solve for inflation rate along BGP, using money supply growth σ,
  - and exchange constraint over two time periods, t, $t+1$,

$$\begin{aligned} M_{t+1} &= P_{t+1}c_{t+1}; \\ M_t &= P_t c_t. \end{aligned}$$

$$\begin{aligned} 1+\sigma_{t+1} &= \frac{M_{t+1}}{M_t} = \frac{P_{t+1}}{P_t}\frac{c_{t+1}}{c_t} = (1+\pi_{t+1})(1+g_{t+1}); \\ g &= 0, \\ &\implies \sigma_t = \pi_t; \end{aligned}$$

$$\begin{aligned} 1+R_t &= (1+\rho)(1+\pi_t) = (1+\rho)(1+\sigma_t); \\ R_t &\simeq \rho+\sigma_t. \end{aligned}$$

- Nominal interest rate sum of time preference, money supply growth.
- Policy change in exogenous σ causes change in R.

- Inflation lowers capital stock,shifts back both AS, AD,
  - increases leisure, decreases employment, output.
- Substitute government budget constraint in consumer budget:

$$c_t^d = w_t l_t^s + r_t k_t - k_{t+1} + k_t (1 - \delta_k).$$

  - Along the BGP, $k_{t+1} = k_t$, $r_t - \delta_k = \rho$,

$$\begin{aligned} c_t^d &= w_t l_t^s + \rho k_t, \; l_t^s = 1 - x_t, \; x_t = \frac{\alpha c_t (1 + R_t)}{w_t}, \\ c_t^d &= \frac{1}{1 + \alpha (1 + R_t)} (w_t + \rho k_t), \; y_P \equiv w_t + \rho k_t, \\ c_t^d &= \frac{y_P}{1 + \alpha (1 + R_t)} \end{aligned}$$

  - Consumed share of permanent income falls as σ_t, R_t rise.
  - Share no longer constant, but decreased by inflation tax.

- Aggregate demand AD : sum of consumption demand, investment:

$$
\begin{aligned}
y_t^d &= \left(\frac{1}{1+\alpha\left(1+R_t\right)}\right)\left(w_t+\rho k_t\right)+\delta_k k_t, \\
&= \left(\frac{1}{1+\alpha\left(1+R_t\right)}\right)\left[w_t+\left(\rho+\delta_k\left[1+\alpha\left(1+R_t\right)\right]\right)k_t\right]; \\
\frac{1}{w_t} &= \frac{1}{y_t^d\left[1+\alpha\left(1+R_t\right)\right]-\left(\rho+\delta_k\left[1+\alpha\left(1+R_t\right)\right]\right)k_t}.
\end{aligned}
$$

- AS unchanged:

$$
\frac{1}{w_t}=\frac{\left(y_t^s\right)^{\frac{1-\gamma}{\gamma}}}{\gamma A_G^{\frac{1}{\gamma}}\left(k_t\right)^{\frac{1-\gamma}{\gamma}}}
$$

Example 20.3: Solution for Capital Stock

- As in Example 8.1 : assume $\rho = 0.03$, $\gamma = \frac{1}{3}$, $\delta_k = 0.03$, $\alpha = 0.5$, $A_G = 0.15$;
  - assume money supply growth rate $\sigma = 5\%$,
  - $R = \rho + \sigma + \rho\sigma = 0.03 + 0.05 + 0.03\,(0.05) = 0.0815$,
  - capital stock $k_t = 2.2242$, while $k_t = 2.3148$ when $R = 0$.
- Solution for k_t as in Chapter 10 methodology,
  - except everywhere that α appears, now is $a\,(1+R)$:

$$
\begin{aligned}
k_t &= \frac{T\gamma A_G^{\frac{1}{\gamma}} \left[\frac{(1-\gamma)}{\rho+\delta_k}\right]^{\frac{1-\gamma}{\gamma}}}{\left(\gamma + \alpha\,(1+R)\right)\left(\frac{\rho+\delta_k}{(1-\gamma)}\right) - \alpha\,(1+R)\,\delta_k} \\
&= \frac{\left(\frac{1}{3}\right)(0.15)^3\left(\frac{2}{3(0.06)}\right)^2}{\left(\frac{1}{3} + 0.5\,(1.0815)\right)(0.06)\,(1.5) - 0.5\,(1.0815)\,(0.03)} \\
&= 2.2242.
\end{aligned}
$$

Example 20.3 AS-AD

$$\frac{1}{w_t} = \frac{\left(y_t^s\right)^{\frac{1-\gamma}{\gamma}}}{\gamma A_G^{\frac{1}{\gamma}}\left(k_t\right)^{\frac{1-\gamma}{\gamma}}} = \frac{\left(y_t^s\right)^2}{0.333\left(0.15\right)^3\left(2.2242\right)^2};$$

$$\frac{1}{w_t} = \frac{1}{y_t^d\left[1+\alpha\left(1+R_t\right)\right]-k_t\left(\rho+\delta_k\left[1+\alpha\left(1+R_t\right)\right]\right)}$$

$$= \frac{1}{y_t^d\left(1+0.5\left(1.0815\right)\right)-2.2\left(0.03+\left(0.03\right)\left[1+0.5\left(1.0815\right)\right]\right)}.$$

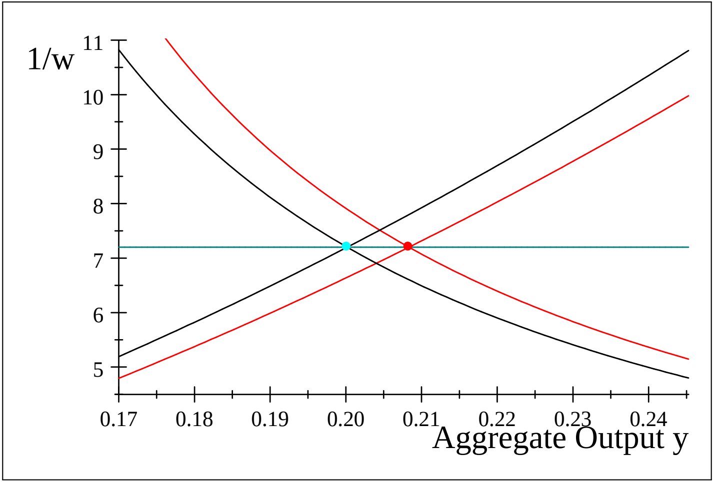


Figure 20.1. Monetary Economy $AS - AD$ with Inflation Tax (black) in Example 20.3 and No Tax (red) in Example 8.1.

Consumption and output fall from inflation tax:

$$c_t^d = \frac{w_t + \rho k_t}{1 + \alpha\left(1 + R_t\right)} = \frac{0.1389 + 0.03\left(2.2242\right)}{1 + 0.5\left(1 + 0.0815\right)} = 0.133\,46;$$

$$y_t^d = \frac{w_t + \left(\rho + \delta_k\left[1 + \alpha\left(1 + R_t\right)\right]\right) k_t}{1 + \alpha\left(1 + R_t\right)} = \frac{0.139 + \left(0.03 + 0.03\left(1 + 0.5\left(1.0815\right)\right)\right)\left(2.2\right)}{1 + 0.5\left(1.0815\right)} = 0.200\,18.$$

Consumption to output ratio unaffected by inflation tax:

$$\frac{c_t^d}{y_t^d} = \frac{0.133\,46}{0.200\,18} = 0.667.$$

Labor Market Effects of Inflation

- Labor demand same as Chapter 8 :

$$l_t^d = \left(\frac{\gamma A_G}{w_t}\right)^{\frac{1}{1-\gamma}} k_t;$$

- Labor supply decreased:

$$\begin{aligned} l_t^s &= 1 - x_t,\ x_t = \frac{\alpha c_t \left(1 + R_t\right)}{w_t},\ c_t = \frac{\left(w_t + \rho k_t\right)}{1 + \alpha\left(1 + R_t\right)} \\ l_t^s &= 1 - \frac{\alpha c_t \left(1 + R_t\right)}{w_t} \\ &= 1 - \frac{\alpha\left(1 + R_t\right)\left(1 + \frac{\rho k_t}{w_t}\right)}{\left[1 + \alpha\left(1 + R_t\right)\right]}. \end{aligned}$$

  - Higher R_t causes lower labor supply, even as k_t decreases.

Inverted labor supply and demand, calibrated, for graphing

$$w_t = \gamma A_G \left(\frac{k_t}{l_t^d}\right)^{1-\gamma} = \frac{(0.15)}{3}\left(\frac{(2.2242)}{l_t^d}\right)^{\frac{2}{3}}.$$

$$\begin{aligned} w_t &= \frac{\alpha(1+R_t)\rho k_t}{(1-l_t^s)(1+\alpha(1+R_t))-\alpha(1+R_t)} \\ &= \frac{0.5(1.0815)(0.03)(2.2242)}{(1-l_t^s)(1+0.5(1.0815))-0.5(1.0815)}. \end{aligned}$$

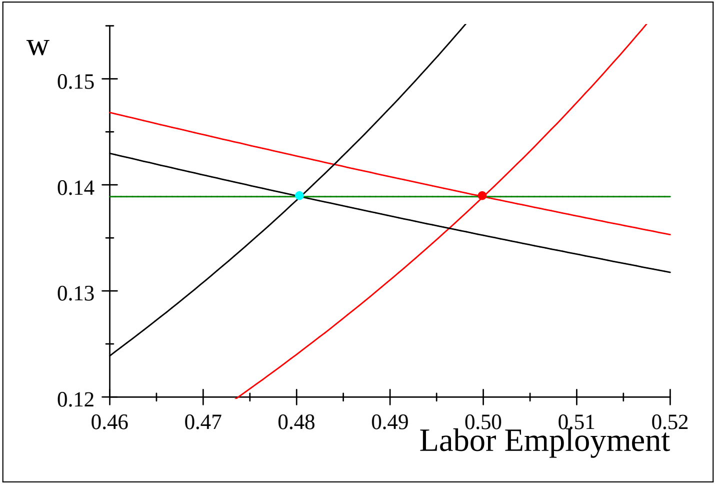


Figure 20.2. Zero Growth Equilibrium Labor Market with Inflation Tax of $R = 0.0815$ (black) in Example 20.3 Compared to $R = 0$ (red).

Employment Effect of Inflation Tax

- Wage unchanged;
- labor supply and demand shift back;
- employment falls:

$$l_t^s = 1 - \frac{0.5\,(1.0815)\left(1 + \frac{0.03(2.2242)}{0.13889}\right)}{(1 + 0.5\,(1.0815))} = 0.480\,42.$$

  - compared to $l_t^s = 0.5$ with $R_t = 0$;
  - an employment fall of $\frac{0.5 - 0.480\,42}{0.5} = 0.039$, or 3.9%.

Isocost, Isoquant, Factor Ratio with Inflation Tax

- $y = 0.200\,31,\ w = 0.25071,\ r = 0.11526:$
  - Isocost line

$$
\begin{aligned}
0.20018 &= y_t = w_t l_t + r_t k_t = (0.13889)\, l_t + (0.06)\, k_t, \\
k_t &= \frac{0.20018}{0.06} - \frac{(0.13889)\, l_t}{0.06}.
\end{aligned}
$$

  - Isoquant curve

$$
\begin{aligned}
0.20018 &= y_t^s = A_G \left(l_t^d h_t\right)^{\gamma} (k_t)^{1-\gamma} = (0.15) \left(l_t^d h_t\right)^{\frac{1}{3}} (k_t)^{\frac{2}{3}}, \\
k_t &= \left(\frac{0.20018}{(0.15)\left(l_t^d\right)^{\frac{1}{3}}}\right)^{\frac{3}{2}} = \frac{\left(\frac{0.20018}{0.15}\right)^{\frac{3}{2}}}{\left(l_t^d\right)^{\frac{1}{2}}}.
\end{aligned}
$$

Factor input ratio

$$
\frac{k_t}{l_t} = \frac{2.2242}{0.480\,42} = 4.629\,7.
$$

Same Factor Ratio, Lower Output, Employment

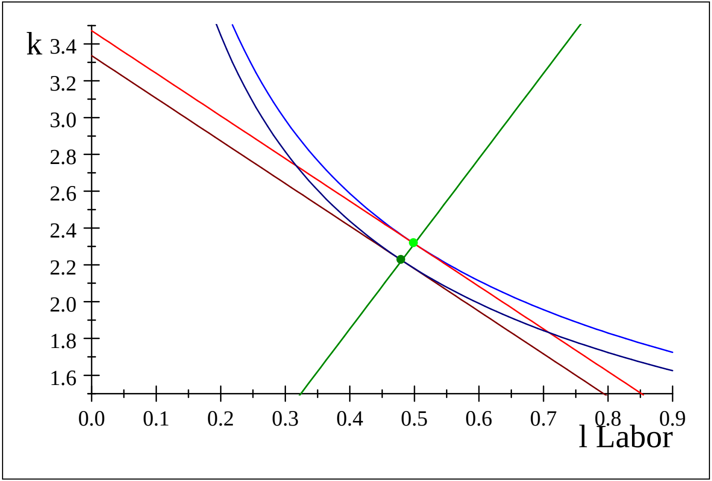


Figure 20.3. Factor Market Equilibrium in Monetary Economy with $R = 0.0815$ in Example 20.3 (darker red, blue) and $R = 0$ (lighter red, blue).

Production, Utility, Budget Line with Inflation Tax

Production

$$
\begin{aligned}
c_t^d &= y_t^s - i_t = A_G \left(l_t^d\right)^{\gamma} (k_t)^{1-\gamma} - \delta_k k_t, \\
c_t^d &= (0.15) \left(l_t^d\right)^{\frac{1}{3}} (2.2242)^{\frac{2}{3}} - (0.03)(2.2242)
\end{aligned}
$$

Utility

$$
\begin{aligned}
-2.3413 &= u = \ln c_t + \alpha \ln x_t = \ln c_t + 0.5 \ln (1 - l_t), \\
c_t &= \frac{e^{-2.3413}}{(1 - l_t)^{0.5}}.
\end{aligned}
$$

Budget line

$$
c_t^d = w_t l_t^s + \rho k_t^s = (0.13889)\, l_t^s + (0.03)(2.2242).
$$

Inflation Decreases Consumption, Employment

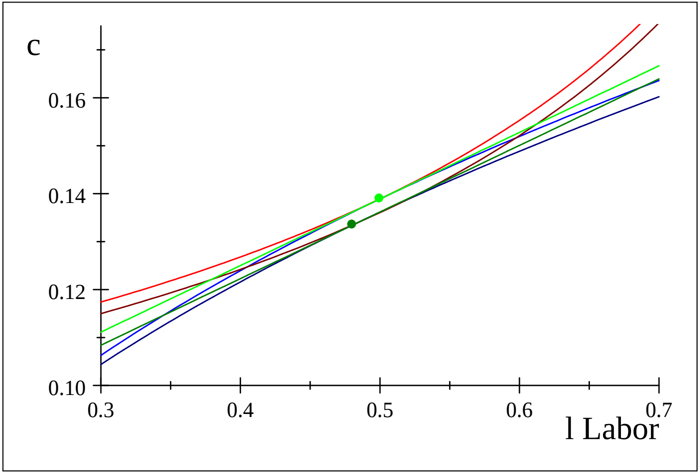


Figure 20.4. General Equilibrium Consumption and Utility Levels with Inflation Tax of $R = 0.0815$ in Example 20.3 (darker red, blue, green), and Baseline $R = 0$ (lighter red, blue, green).

Endogenous Growth and the Inflation Tax

- Inflation effect on growth similar to Chapter 19.
  - affects goods-leisure margin, as consumption tax, labor income tax.
  - comparison of taxes within margin implies distortion equivalence:

$$1 + \tau_c = \frac{1}{1 - \tau_l} = 1 + R.$$

  - $R = 0.10$: same as $\tau_c = 0.10$, or $\tau_l = \frac{\tau_c}{1+\tau_c} = \frac{0.10}{1.10} = 0.091$.

Constraints of Endogenous Growth Monetary Economy

- Endogenous growth economy combined with monetary economy:

$$
\begin{aligned}
m_t &= c_t, \\
c_t &= w_t l_t h_t + r_t k_t - k_{t+1} + k_t (1 - \delta_k) - b_{t+1} (1 + \pi_{t+1}) \\
&\quad + b_t (1 + R_t) - m_{t+1} (1 + \pi_{t+1}) + m_t + \frac{G_t}{P_t}, \\
l_t &= \frac{-r_t k_t + k_{t+1} - k_t (1 - \delta_k) + \frac{G_t}{P_t}}{w_t h_t} \\
&\quad - \frac{b_{t+1} (1 + \pi_{t+1}) - b_t (1 + R_t) + m_{t+1} (1 + \pi_{t+1})}{w_t h_t}, \\
x_t &= 1 - l_{Ht} - l_t.
\end{aligned}
$$

- Government budget constraint the same as exogenous growth

$$
\frac{G_t}{P_t} = b_{t+1} (1 + \pi_{t+1}) - b_t (1 + R_t) + m_{t+1} (1 + \pi_{t+1}) - m_t.
$$

Consumer Problem with Endogenous Growth, Money

$$
\begin{aligned}
& V\left(k_t, h_t, b_t, m_t\right) \\
= \quad & \underset{k_{t+1}, b_{t+1}, m_{t+1}, l_{Ht}}{Max} : \ln\left(m_t\right) + \alpha \ln\left[1 - l_{Ht} - \frac{-r_t k_t + k_{t+1} + \frac{G_t}{P_t}}{w_t h_t}\right. \\
& \left. - \frac{-k_t\left(1-\delta_k\right) + b_{t+1}\left(1+\pi_{t+1}\right) - b_t\left(1+R_t\right) + m_{t+1}\left(1+\pi_{t+1}\right)}{w_t h_t}\right] \\
& + \beta V\left(k_{t+1}, h_t\left(1-\delta_h\right) + A_H l_{Ht} h_t, b_{t+1}, m_{t+1}\right);
\end{aligned}
$$

$$
\Longrightarrow
$$

$$
x_t = \frac{\alpha\left(1+R_t\right) c_t}{w_t h_t}.
$$

- Same intratemporal condition as in
  - exogenous growth monetary economy, plus human capital h_t.
- Same intertemporal conditions as in Chapter 12

AS-AD Analysis: Consumption Demand

$$
\begin{aligned}
c_t^d &= w_t h_t l_t^s + r_t k_t - k_{t+1} + k_t (1 - \delta_k), \\
c_t^d &= w_t h_t (1 - x_t - l_{Ht}) + r_t k_t - k_{t+1} + k_t (1 - \delta_k), \\
c_t^d &= w_t h_t \left(1 - \frac{\alpha (1 + R_t) c_t}{w_t h_t} - l_{Ht} \right) \\
&\quad + r_t k_t - k_{t+1} + k_t (1 - \delta_k), \\
c_t^d [1 + \alpha (1 + R_t)] &= w_t h_t (1 - l_{Ht}) + r_t k_t - k_{t+1} + k_t (1 - \delta_k), \\
c_t^d &= \frac{w_t h_t (1 - l_{Ht}) + r_t k_t - k_{t+1} + k_t (1 - \delta_k)}{[1 + \alpha (1 + R_t)]}, \\
c_t^d &= \frac{w_t h_t \left(1 - \frac{g + \delta_H}{A_H} \right) + \rho (1 + g) k_t}{[1 + \alpha (1 + R_t)]}.
\end{aligned}
$$

Adding Investment, AS-AD

- Investment along BGP:

$$i_t = k_t \left(g + \delta_k \right),$$

  - Aggregate output: $y_t^d = c_t^d + i_t$

$$y_t^d = \frac{w_t h_t \left(1 - \frac{g+\delta_H}{A_H}\right) + \rho \left(1+g\right) k_t}{\left[1 + \alpha \left(1 + R_t\right)\right]} + k_t \left(g + \delta_k\right);$$

$$\frac{1}{w_t} = \frac{\left(1 - \frac{g+\delta_H}{A_H}\right)}{\frac{y_t^d}{h_t}\left[1 + \alpha\left(1+R_t\right)\right] - \frac{k_t}{h_t}\left[\rho\left(1+g\right) + \left(g+\delta_k\right)\left[1+\alpha\left(1+R_t\right)\right]\right]}$$

  - Aggregate supply unchanged:

$$y_t^s = A_G \left(\frac{\gamma A_G}{w_t}\right)^{\frac{\gamma}{1-\gamma}} k_t;$$

$$\frac{1}{w_t} = \frac{\left(y_t^s\right)^{\frac{1-\gamma}{\gamma}}}{\gamma A_G^{\frac{1}{\gamma}} \left(k_t\right)^{\frac{1-\gamma}{\gamma}}}.$$

Solution Methodology: Zero Excess Output Demand

- Excess aggregate output demand function

$$Y\left(w_t, h_t, k_t, g\right) = y_t^d - y_t^s = 0,$$

$$0 = \frac{w_t h_t \left(1 - \frac{g+\delta_H}{A_H}\right) + k_t \left[\rho\left(1+g\right) + \left(g+\delta_k\right)\left[1+\alpha\left(1+R_t\right)\right]\right]}{\left[1+\alpha\left(1+R_t\right)\right]}$$
$$-A_G \left(\frac{\gamma A_G}{w_t}\right)^{\frac{\gamma}{1-\gamma}} k_t.$$

  - Dividing through by $w_t h_t$

$$0 = \frac{\left(1 - \frac{g+\delta_H}{A_H}\right) + \frac{k_t}{w_t h_t}\left[\rho\left(1+g\right) + \left(g+\delta_k\right)\left[1+\alpha\left(1+R_t\right)\right]\right]}{\left[1+\alpha\left(1+R_t\right)\right]}$$
$$-A_G \left(\frac{\gamma A_G}{w_t}\right)^{\frac{\gamma}{1-\gamma}} \frac{k_t}{w_t h_t}.$$

  - Solve for $\frac{k_t}{w_t h_t}$ and w_t in terms of g : one equation in only g.

Solve for Wage Rate in Terms of g

$$
\begin{aligned}
l_t &= \frac{(1+g)(1-\beta)}{A_H \beta}; \\
r_t &= (1-\gamma) A_G \left(\frac{k_t}{h_t l_t} \right)^{-\gamma}, \\
\frac{k_t}{h_t l_t} &= \left(\frac{(1-\gamma) A_G}{r_t} \right)^{\frac{1}{\gamma}}, \\
\frac{k_t}{h_t l_t} &= \left(\frac{(1-\gamma) A_G}{(1+g)(1+\rho) - 1 + \delta_k} \right)^{\frac{1}{\gamma}}; \\
w_t &= \gamma A_G \left(\frac{k_t}{h_t l_t} \right)^{1-\gamma}, \\
w_t &= \gamma A_G \left(\frac{(1-\gamma) A_G}{(1+g)(1+\rho) - 1 + \delta_k} \right)^{\frac{1-\gamma}{\gamma}}.
\end{aligned}
$$

$$\frac{k_t}{w_t h_t} = \left(\frac{k_t}{h_t l_t}\right)\frac{l_t}{w_t} =$$

$$\left(\frac{(1-\gamma)A_G}{(1+g)(1+\rho)-1+\delta_k}\right)^{\frac{1}{\gamma}} \frac{\frac{(1+g)(1-\beta)}{A_H\beta}}{\gamma A_G\left(\frac{(1-\gamma)A_G}{(1+g)(1+\rho)-1+\delta_k}\right)^{\frac{1-\gamma}{\gamma}}}$$

$$\frac{k_t}{w_t h_t} = \left(\frac{(1-\gamma)}{(1+g)(1+\rho)-1+\delta_k}\right)\frac{(1+g)(1-\beta)}{\gamma A_H \beta}.$$

- Substitute in $\frac{k_t}{w_t h_t}$ and w_t:

$$0 = \frac{\left(1 - \frac{g+\delta_H}{A_H}\right) + \frac{(1-\gamma)(1+g)(1-\beta)[\rho(1+g)+(g+\delta_k)[1+\alpha(1+R_t)]]}{[(1+g)(1+\rho)-1+\delta_k]\gamma A_H \beta}}{[1+\alpha(1+R_t)]} - \frac{A_G}{\left(\frac{(1-\gamma)A_G}{(1+g)(1+\rho)-1+\delta_k}\right)} \frac{(1-\gamma)(1+g)(1-\beta)}{\gamma A_H \beta [(1+g)(1+\rho)-1+\delta_k]}.$$

- This reduces to

$$0 = \frac{\left(1 - \frac{g+\delta_H}{A_H}\right) + \frac{(1-\gamma)(1+g)(1-\beta)[\rho(1+g)+(g+\delta_k)[1+\alpha(1+R_t)]]}{[(1+g)(1+\rho)-1+\delta_k]\gamma A_H \beta}}{[1+\alpha(1+R_t)]} - \frac{(1+g)(1-\beta)}{\gamma A_H \beta}.$$

$$\begin{aligned} 0 = {} & \beta\gamma(A_H - g - \delta_H)[(1+g)+\beta(\delta_k - 1)] \\ & + \beta(1-\gamma)(1+g)(1-\beta)[\rho(1+g)+(g+\delta_k)[1+\alpha(1+R_t) \\ & -(1+g)(1-\beta)[(1+g)+\beta(\delta_k - 1)][1+\alpha(1+R_t)] \end{aligned}$$

Example 20.4 Inflation Tax and Growth

- Assume same parameters as in Example 13.2,
  - $\gamma = \frac{1}{3}$, $\alpha = 1$, $A_h = 0.20$, $\delta_k = 0.05$, $\delta_h = 0.015$, $\beta = \frac{1}{1+\rho} = 0.95$, $A_G = 0.282\,24$;
  - assume money supply growth rate of $\sigma = 5\%$:
  - $\Longrightarrow R = \rho + \sigma + \rho\sigma = 0.03 + 0.05 + 0.03\,(0.05) = 0.0815.$
- Equilibrium growth rate given by quadratic equation in g,
  - the excess output demand function.
  - Can graph equation directly to find g.

- $g = 0.02362$, reduction from $g = 0.0333$ when $R = 0$.

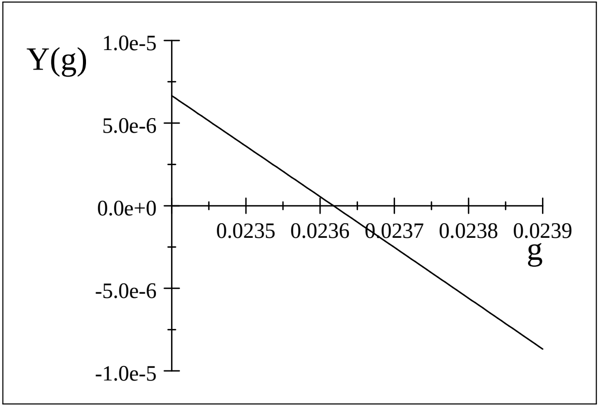


Figure 20.5. Normalized Excess Output Demand $Y(g)$ with Inflation Tax of $R = 0.0815$ in Example 20.4.

Analytic Solution for g

- $Ag^2 + Bg + C = 0,$

$$
\begin{aligned}
A &\equiv -\beta\gamma + \beta(1-\beta)(1-\gamma)[1+\alpha(1+R_t)+\rho] \\
&\quad -[1+\alpha(1+R_t)](1-\beta), \\
B &\equiv -\beta\gamma[1+\beta(\delta_k-1)-A_H+\delta_H] \\
&\quad +\beta(1-\beta)(1-\gamma)\{\rho+\delta_k[1+\alpha(1+R_t)]+1 \\
&\quad +\alpha(1+R_t)+\rho\} - [1+\alpha(1+R_t)](1-\beta)[2+\beta(\delta_k-1)], \\
C &\equiv \beta\gamma(A_H-\delta_H)[1+\beta(\delta_k-1)] \\
&\quad +\beta(1-\beta)(1-\gamma)\{\rho+\delta_k[1+\alpha(1+R_t)]\} \\
&\quad -[1+\alpha(1+R_t)](1-\beta)[1+\beta(\delta_k-1)],
\end{aligned}
$$

$$
\begin{aligned}
A &= -0.3532, B = -0.0140, C = 0.0005269; \\
g &= 0.0236.
\end{aligned}
$$

Solution for Capital Ratio Needed for AS-AD

- Given $g = 0.02362$, can find all other variables, including $\frac{k_t}{h_t}$:

$$l_t = \frac{(1+g)(1-\beta)}{A_H \beta} = \frac{(1+0.02362)(1-0.95)}{(0.20)\,0.95} = 0.26937;$$

$$\frac{k_t}{h_t l_t} = \left(\frac{(1-\gamma) A_G}{(1+g)(1+\rho) - 1 + \delta_k} \right)^{\frac{1}{\gamma}},$$

$$\frac{k_t}{h_t l_t} = \left(\frac{\left(\frac{2}{3}\right) 0.282\,24}{(1+0.02362)(1+0.052632) - 1 + 0.05} \right)^{3} = 3.2144;$$

$$\frac{k_t}{h_t} = \left(\frac{k_t}{h_t l_t} \right) l_t = (3.214\,4)(0.26937)$$

$$= 0.86586.$$

- Capital ratio rises to 0.86586 compared to 0.694 when $R = 0$.
  - As with labor tax: inducing substitution from human to physical capital.

Calibrated AS and AD Functions

$$\frac{1}{w_t} = \frac{\left(1 - \frac{0.02362+0.015}{0.20}\right)}{y\,(2.0815) - (0.86586)\left[0.0526\,(1.0236) + (0.0736)\,(2.0815)\right]};$$

$$\frac{1}{w_t} = \frac{3}{0.282\,24}\left(\frac{1}{0.282\,24\,(0.86586)}\right)^2 y_t^2.$$

AD, AS Both Shift Out as Compared to Baseline R=0

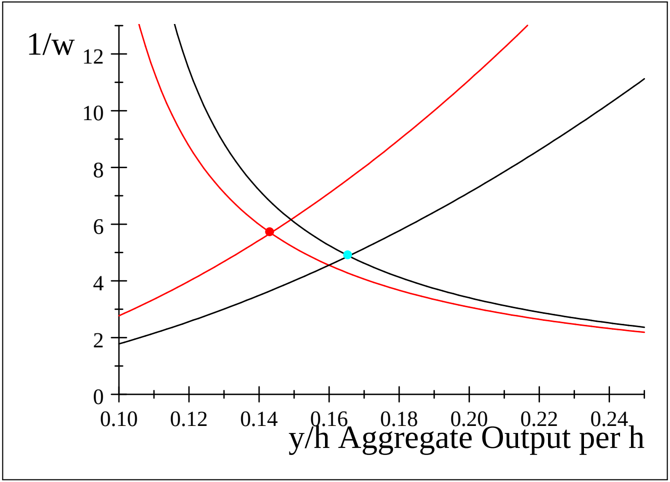


Figure 20.6. $AS - AD$ with Endogenous Growth and an Inflation Tax of $R = 0.0815$ in Example 20.4 (black) versus Baseline $R = 0$ in Example 13.2 (red)

Effect on Wage Rate

- Exact wage is $w_t = 0.20491$;
  - relative output price $\frac{1}{w_t} = \frac{1}{0.204\,91} = 4.8802.$
  - Wage rate up from $R = 0$ rate of $w_t = 0.1757.$

$$w_t = \gamma A_G \left(\frac{k_t}{h_t l_t}\right)^{1-\gamma} = \frac{1}{3}(0.28224)(3.214\,4)^{\frac{2}{3}};$$
$$w_t = 0.204\,91.$$

Inflation Raises Consumption to Output Ratio

$$\frac{c_t^d}{h_t} = \frac{w_t\left(1 - \frac{g+\delta_H}{A_H}\right) + \rho\left(1+g\right)\frac{k_t}{h_t}}{1+\alpha\left(1+R_t\right)},$$

$$= \frac{0.2049\left(1 - \frac{0.03862}{0.20}\right) + \left(0.0526\right)\left(1.0236\right)\left(0.86586\right)}{\left(1.0815\right)} = 0.10184.$$

$$\frac{y_t^d}{h_t} = \frac{w_t\left(1 - \frac{g+\delta_H}{A_H}\right) + \frac{k_t}{h_t}\left[\rho\left(1+g\right) + \left(g+\delta_k\right)\left(1+\alpha\left(1+R_t\right)\right)\right]}{1+\alpha\left(1+R_t\right)}$$

$$= \frac{0.204\,91\left(1 - \frac{0.02362+0.015}{0.20}\right)}{\left(1+1.0815\right)} + \frac{\left(0.865\,86\right)\left(0.0526\right)\left(1.02362\right)}{\left(1.0815\right)}$$

$$+\frac{\left(0.865\,86\right)\left(0.02362+0.05\right)\left(1.0815\right)}{\left(1.0815\right)} = 0.16559;$$

$$\frac{c_t^d}{y_t^d} = \frac{0.10184}{0.16559} = 0.615;\ R = 0 : \frac{c_t^d}{y_t^d} = 0.597.$$

Interest Rate

- Real interest r_t falls because of decrease in growth rate to 12.75%
  - from 13.8% with $R = 0$.

$$\begin{aligned} r_t &= (1+g)(1+\rho) - 1 + \delta_k, \\ r_t &= (1+0.02362)(1+0.052632) - 1 + 0.05 = 0.12750. \end{aligned}$$

- Inflation rate of 0.37%, less than 1%, given by Fisher equation

$$\begin{aligned} 1 + R_t &= (1 + r_t - \delta_k)(1+\pi_t), \\ (1+\pi_t) &= \frac{1.0815}{(1.12750 - 0.05)} = 1.0037. \end{aligned}$$

- Nominal interest rate is $R = 0.0815$, or 8.15%.
  - Implies an inflation tax rate of 8.15%,
  - even though inflation only 0.37%.

Government Spending as a Fraction of Output

- Assuming government bonds equal to zero, along BGP
  - real government expenditure given by inflation tax proceeds:

$$\frac{G_t}{P_t} = \frac{M_{t+1} - M_t}{P_t} = \frac{M_{t+1}\left(\frac{P_{t+1}}{P_t}\right)}{P_{t+1}P_t} - \frac{M_t}{P_t},$$
$$= m_t\,(1+g)\,(1+\pi_t) - m_t;$$
$$\frac{G_t}{P_t y_t} = \frac{m_t\left[(1+g)\,(1+\pi_t) - 1\right]}{y_t} = \frac{m_t}{c_t}\frac{c_t}{y_t}\left[(1+g)\,(1+\pi_t) - 1\right]$$
$$= \frac{0.10184}{0.16559}\left[(1.02362)\,(1.0037) - 1\right] = 0.016856.$$

  - Inflation tax raises revenue of 1.7% of output.
- Present discounted value of inflation tax revenue relative to output:

$$\frac{\frac{G_t}{P_t y_t}}{\rho\,(1+g)} = \frac{0.016856}{0.052632\,(1.02362)} = 0.31287.$$

  - Equals 31% of current output.

Constructing Labor Suppy; Same Labor Demand

$$
\begin{aligned}
x_t &= \frac{c_t^d \alpha (1 + R_t)}{w_t h_t}, \\
l_t^s &= 1 - \frac{c_t^d \alpha (1 + R_t)}{w_t h_t} - l_{Ht}. \\
c_t^d &= \frac{w_t h_t \left(1 - \frac{g + \delta_H}{A_H}\right) + \rho (1 + g) k_t}{1 + \alpha (1 + R_t)}, \\
l_t^s &= 1 - \left(\frac{\alpha (1 + R_t)}{1 + \alpha (1 + R_t)}\right) \left[\left(1 - \frac{g + \delta_H}{A_H}\right) + \frac{\rho (1 + g) k_t}{w_t h_t}\right] - \frac{g + \delta_H}{A_H}, \\
l_t^s &= \frac{\left(1 - \frac{g + \delta_H}{A_H}\right) - \frac{\alpha (1 + R_t) \rho (1 + g) k_t}{w_t h_t}}{1 + \alpha (1 + R_t)}.
\end{aligned}
$$

$$
l_t^d = \left(\frac{\gamma A_G}{w_t}\right)^{\frac{1}{1-\gamma}} \frac{k_t}{h_t}.
$$

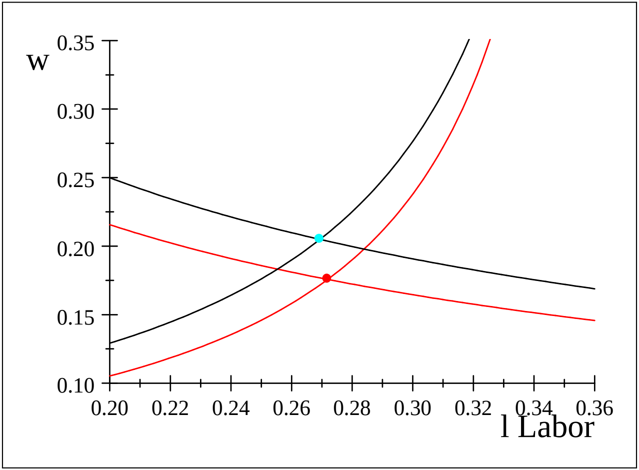
Inverted and Calibrated Labor Supply, Demand

$$
\begin{aligned}
w_t &= \frac{\alpha\left(1+R_t\right)\rho\left(1+g\right)\frac{k_t}{h_t}}{\left(1-\frac{g+\delta_H}{A_H}\right)-l_t^s\left(1+\alpha\left(1+R_t\right)\right)};\\
w_t &= \gamma A_G\left(\frac{k_t}{h_t l_t^d}\right)^{1-\gamma}.
\end{aligned}
$$

$$
\begin{aligned}
w_t &= \frac{1.0815\left(0.052632\right)\left(1+0.02362\right)\left(0.865\,86\right)}{\left(1-\frac{(0.02362+0.015)}{0.20}\right)-l_t^s\left(1+1.0815\right)};\\
w_t &= \frac{\left(0.282\,24\right)}{3}\frac{\left(0.865\,86\right)^{\frac{2}{3}}}{\left(l_t^d\right)^{\frac{2}{3}}}.
\end{aligned}
$$

Inflation Shifts Up Labor Supply, Demand,

Decreases Employment, Raises Real Wage



Fitgure 20.7. Labor Market with Endogenous Growth Inflation Tax of $R = 0.0815$ in Example 20.4 (black) and $R = 0$ in Example 13.2 (red).

Isocost, Isoquant, Factor Ratio with Inflation Tax

Isocost line

$$
\begin{aligned}
0.16559 &= y_t = w_t l_t h_t + r_t k_t = (0.20491)\, l_t h_t + (0.1275)\, k_t, \\
\frac{k_t}{h_t} &= \frac{0.165\,59}{0.1275 h_t} - \frac{(0.20491)\, l_t}{0.1275}.
\end{aligned}
$$

Isoquant curve

$$
\begin{aligned}
0.16559 &= y_t^s = A_G \left(l_t^d h_t \right)^{\gamma} (k_t)^{1-\gamma} = (0.28224) \left(l_t^d h_t \right)^{\frac{1}{3}} (k_t)^{\frac{2}{3}}, \\
\frac{k_t}{h_t} &= \left(\frac{0.165\,59}{(0.28224)\, h_t \left(l_t^d \right)^{\frac{1}{3}}} \right)^{\frac{3}{2}} = \frac{\left(\frac{0.165\,59}{(0.28224) h_t} \right)^{\frac{3}{2}}}{\left(l_t^d \right)^{\frac{1}{2}}}.
\end{aligned}
$$

Factor input ratio

$$
\frac{k_t}{h_t l_t} = \frac{0.865\,86}{0.269\,37} = 3.2144.
$$

Inflation Raises Factor Input Ratio, Lowers Employment

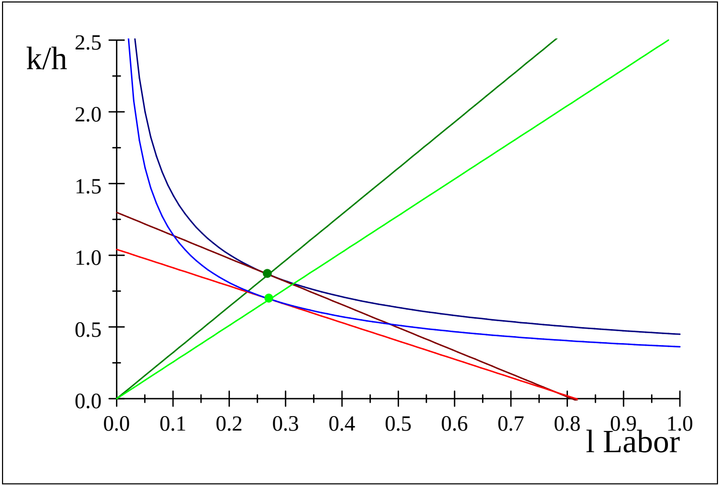


Figure 20.8. Factor Market Equilibrium with Endogenous Growth in Example 20.4 with $R = 0.0815$ (darker red, blue, green), Compared to $R = 0$ (lighter red, blue, green).

Production, Utility Level, Budget Line

Production

$$c_t^d = y_t^s - i_t = A_G \left(l_t^d h_t \right)^{\gamma} (k_t)^{1-\gamma} - (g + \delta_k)\, k_t,$$

$$\frac{c_t^d}{h_t} = (0.28224) \left(l_t^d \right)^{\frac{1}{3}} (0.865\,86)^{\frac{2}{3}} - (0.0736)\,(0.866)\,.$$

Utility Level

$$-2.9051 = u = \ln c_t + \alpha \ln x_t = \ln c_t + \alpha \ln (1 - l_{Ht} - l_t)\,,$$

$$= \ln 0.10184 + 1 \ln (1 - 0.1931 - 0.26937)\,,$$

$$c_t = \frac{e^{-2.9051}}{(1 - 0.193\,1 - l_t)}.$$

Budget line

$$c_t^d = w_t l_t^s h_t + k_t \left[r_t - (g + \delta_k) \right],$$

$$\frac{c_t^d}{h_t} = (0.204\,91)\, l_t^s + (0.865\,86)\,(0.127\,50 - 0.05 - 0.02362)\,.$$

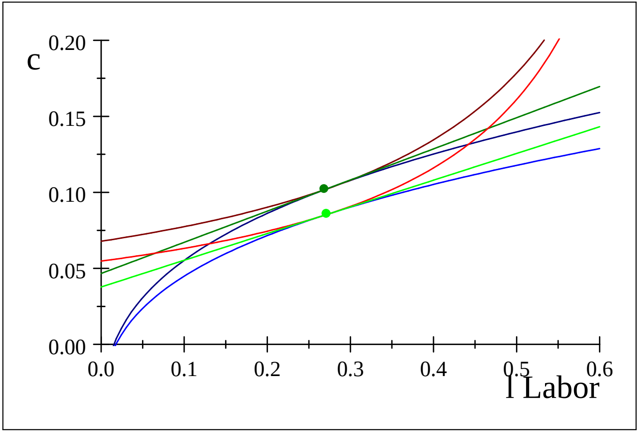


Figure 20.9. General Equilibrium with Endogenous Growth and an Inflation Tax of $R = 0.0815$ in Example 20.4 (darker red, blue, green) Compared to $R = 0$ (lighter red, blue, green) as in Example 13.2.

Growth Effect of Inflation

- Inflation tax, through R, has similar growth effects to other taxes.
  - Consistent with evidence
  - of negative effect of inflation on output growth.
  - Partly why inflation targeting policy seeks low inflation rate.
- Optimal policy: inflation tax compounds composite labor tax,
  - as does goods tax, causing substitution from goods to labor.
  - Also: substitution from money use towards credit use for exchange.
- Producing exchange credit to avoid inflation tax
  - uses up real banking resources, decreases government revenue.
- Research finds inflation tax should be eliminated completely,
  - again implying low inflation targeting policy used widely today.

Exogenous Growth Versus Endogenous Growth

- Inflation tax economy:
  - differences between exogenous growth, endogenous growth
  - similar to comparison for labor tax in Chapters 9, 19.
- Inflation tax causes a smaller decline in employment
  - in endogenous growth than in exogenous growth.
  - Makes endogenous growth closer to data,
  - since moderate inflation causes only
  - moderate decrease in long term employment.

Tobin Effect and Policy of Inflation Tax

- Endogenous growth: wage rises;
  - fixed in exogenous growth.
  - Endogenous growth: substitution from taxed human to untaxed physical capital.
  - Growth decrease causes r_t to fall, w_t to rise.
  - Endogenous growth "Tobin Inflation Effect": $\frac{w}{r}\uparrow$, $\frac{k}{h}\downarrow$.
- Policy implications:
  - Output per h rises in endogenous growth, growth rate falls:
  - Not good policy, though $\frac{y}{h}$ rises, since h can be falling.
  - Consumer pushed from optimal physical versus human capital use.
  - Manifestation of inefficiency of tax,
  - including decline in employment.

Level of the Inflation Tax

- Inflation tax equals llevel of nominal interest rate R.
  - Can be confusing since may think π, inflation rate,
  - is level of inflation tax.
- See why R is level of inflation tax
  - through marginal rate of substitution between goods, leisure.

$$\frac{x_t}{\alpha c_t} = \frac{1 + R_t}{w_t h_t}.$$

  - $1 + R$ is shadow price of goods,
  - now including exchange cost R.
  - Can set σ money supply growth so $R = 0$.
  - Then shadow price of goods is 1 as in no-tax economy.

Optimal Inflation Rate

- $R = 0$, is "first-best" Friedman 1969 optimum.
  - By Fisher equation, optimum implies negative inflation rate:

$$\begin{aligned} R &= 0 = \rho + \pi, \\ \pi &= -\rho. \end{aligned}$$

  - money earns same real return as physical capital, $r = \rho$.
- Optimum of $R = \rho$, with zero inflation, $\pi = 0$,
  - if hold gold reserves in central bank
  - as implicit backing of "fiat" paper money.

Central Bank Policy: Combining Fisher, Intertemporal

- Fisher equation: substitute in for $(1 + r_t - \delta_k)$ from intertemporal margin;

$$
\begin{aligned}
1 + R_t &= (1 + \pi_t)(1 + r_t - \delta_k), \\
1 + g_t &= \frac{1 + r_t - \delta_k}{1 + \rho}, \\
1 + r_t - \delta_k &= (1 + g_t)(1 + \rho), \\
1 + R_t &= (1 + \pi_t)(1 + g_t)(1 + \rho); \\
\ln(1 + x) &\simeq x; \\
R_t &\simeq \rho + \pi_t + g_t.
\end{aligned}
$$

Deriving a Taylor Condition

- Write inflation at time t as "expected inflation" $E_{t-1}\pi_t \equiv \pi_t^e = \pi_t$,
  - with perfect foresight assumption.
  - Define "inflation target" π^*; add, subtract:

$$R_t \simeq \rho + \pi^* + (\pi_t^e - \pi^*) + g_t.$$

- Use more general "constant relative risk aversion" utility
  - $u_t = \frac{(c_t x_t^{\alpha})^{1-\theta}}{1-\theta}$, where θ is risk aversion; log-utility is special case;
  - equation changed by factoring θ parameter with g_t term:

$$R_t \simeq \rho + \pi^* + (\pi_t^e - \pi^*) + \theta g_t.$$

  - Denote coefficients β_π, β_g on inflation term $(\pi_t^e - \pi^*)$, growth g_t:

$$R_t \simeq \rho + \pi^* + \beta_\pi (\pi_t^e - \pi^*) + \beta_g g_t.$$

  - "Taylor condition" of endogenous growth monetary economy.

Equivalence of Taylor Condition, Rule

- Taylor 1993 rule

$$R_t = \bar{r} + \pi^* + 1.5\left(\pi_t^e - \pi^*\right) + (0.5)\, y_{gap,t}.$$

  - Using β_π, β_g

$$R_t = \bar{r} + \pi^* + \beta_\pi\left(\pi_t^e - \pi^*\right) + \beta_g \cdot y_{gap,t}.$$

- Identical to "Taylor condition"

$$R_t \simeq \rho + \pi^* + \beta_\pi\left(\pi_t^e - \pi^*\right) + \beta_g g_t.$$

1. If $\bar{r} = \rho$,
2. $y_{gap,t} \equiv g_t$.

Conditions for Equivalence to Taylor Rule

- $\bar{r}$ defined as historical average real interest rate.
  - Empirically, time preference rate close to such real interest rate.
- Output gap: so-called "speed limit" version of output gap
  - defined exactly as growth rate of output.
- Coefficient β_π :
  - $\beta_\pi = 1$ in model is borderline of Taylor rule "principle": $\beta_\pi > 1$.
  - In BGP equilibrium, Fisher equation holds; $\beta_\pi = 1$.
- Coefficient β_g : degree of relative risk aversion in model.
  - Empirically $\theta \in (0.5, 2.0)$.
  - Taylor 1993 rule assumes $\beta_g = 0.5$, within range of risk aversion.

Taylor Rule as Policy in General Equilibrium

- Taylor condition: equilibrium with government supplying money.
  - Equivalence between money supply rules, interest rate rules.
  - Taylor equation: may say how central bank controls interest rate;
  - may say only how fast it is printing money.
- As equilibrium condition, cannot simply add onto economy,
  - as is usually done for various Taylor rules in many different forms.
  - Rather, exact equilibrium condition; only form consistent with model.
- Endogenous growth model, on BGP, implies simple prescription:
  - while targeting inflation,
  - nominal interest rate should follow economy's output growth rate.
- Simply let R_t follow g_t,
  - as actual inflation equals targeted inflation.

Taylor Condition off BGP Equilibrium

- Endogenous growth model, off BGP, in transition to BGP,
    - implies a dynamic Taylor condition.
    - Employment rate can lag output growth rate,
    - makes employment rate additional factor
    - in dynamic version of Taylor condition.
- Also consumption growth can differ from output growth,
    - and enter dynamic Taylor condition instead of output growth.
- When velocity of money is changing,
    - degree of interest rate "smoothing" enters Taylor condition;
    - velocity change determines "smoothing" parameters.

Taylor Rule in Practice

- "Taylor principle": $\beta_\pi > 1$ rather than $\beta_\pi = 1$ as in Fisher equation.
  - Central bank decides to lower R;
  - by printing money at unexpectedly faster rate,
  - causes real interest rate to fall in short run,
  - causes higher inflation in long run.
- Real interest rate pushed below natural market value
  - by unexpected money supply growth increase:
  - short term subsidy to borrowing in capital markets,
  - followed by long run increase in inflation rate.
  - Adds inflation volatility; raises risk premium, long run real interest rate
- If R raised this way, borrowing is taxed, inflation volatility again rises.

Taylor Rule Subsidization, Taxation of Markets

- "Taylor policy": subsidizing borrowing by lowering real interest rate
  - with subsequent increase in inflation rate.
  - Perhaps to stimulate economy in recession.
- Then when inflation rate does rise,
  - central bank wants to lower inflation rate, taxes borrowing,
  - by raising real interest rate in short run.
- Formula for lack of stable inflation rate,
  - with continuous subsidization or taxation of borrowing.

Lumbering "Ship of State": Capital Market Interference

- Ship-of-state over-steers one way,
  - by subsidizing borrowing, raising inflation rate.
  - Then ship-of-state over-steers other way,
  - by taxing borrowing, lowering inflation rate.
- Inefficient economy: large lumbering ocean supertanker,
  - steered in zigzag pattern around targeted inflation rate,
  - while capital markets alternatively subsidized, taxed.
- Interference with capital markets can lead to financial crises.
  - Too low a real interest causes too much borrowing;
  - cannot be paid back when interest rate rises a lot,
  - when Central Bank confronts rising, high, inflation.

Financial Crises of Last Decade

- Policy implications of Taylor Rule practice:
  - nominal interest rate kept at 1% for 2001-2004;
  - inflation rate eventually increased as money printed,
  - to drive down the nominal interest rate to 1%.
  - Created negative ex-post real interest rate.
- Nominal interest rates allowed to rise when higherinflation;
  - nominal rate jumped up by 5% points.
  - Otherwise real rate would have been kept negative.
- Rise in nominal rate sudden, steeper than expected,
  - consumers defaulted on car, home loans; banks failed.
- Chapter 16 : equivalent to exogenous fall in bank productivity A_F.
  - causing fall in capital stock k_t, drop in stock market, big recession.
- Central bank only controls interest rate by how much money it prints.
  - Causing negative real interest rates: gives root to next crisis.

US Federal Reserve Policy

- Thomas Hoenig, Kansas City Federal Reserve Bank President,
  - describes May 2010: US Federal Reserve policy of last decade:

"So then we come to 2001-2002, and we have language that says we will have low rates for a considerable period, whatever the language was, and we kept interest rates low and real interest rates negative. The decade of the '70s and the decade of the 2000s were the periods in which we had, over 40% of the time, negative real interest rates. The consequence of that was bubbles, high leverage and financial crisis."

Central Bank Policy as Subsidization

- Low interest rate policies: Hoenig 2010 says

  "the saver in America is in a sense subsidizing the borrower in America. We need a more normal set of circumstances so we can have an extended recovery and a more stable economy in the long run....In other words, when we kept interest rates unusually low for a considerable period we favored credit and the allocations related to it over savings, and we created the conditions that I think facilitated a bubble" (O'Grady, Wall Street Journal, 17 May 2010).

More Efficient Policy Possible

- Better would be to allow nominal interest rate
  - to rise and fall with economy's output growth rate.
  - As in Taylor condition along BGP.
- Allows central bank to hit targeted inflation π^*.
  - Keeping inflation constant,
  - only "policy" of central bank:
  - let nominal rate move in exact tandem with growth rate.
- Instead of needing to forecast central bank action,
  - capital markets need only forecast output growth rate,
  - to know how nominal interest rates will change.
- Combine with comprehensive international bank insurance system.